Efficient Steady-state Simulation of High-dimensional Reflected Brownian Motions

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Road map

1. Model Setup and Assumptions
   - Reflected Brownian Motion
   - A Digression: a Naive Euler Scheme
2. Multilevel Monte Carlo Algorithm
   - Assumptions
   - Algorithm Specification
   - Error Bound
3. Numerical Experiments

Reflected Brownian Motion (RBM)

- RBM is the solution of a Skorokhod problem with Brownian input.
- Skorokhod problem:

$$0 \leq \mathbf{Y}(t) = \mathbf{Y}(0) + \mathbf{X}(t) + R\mathbf{L}(t), \ \mathbf{L}(0) = 0 \tag{1}$$

  where the i-th entry of $\mathbf{L}(\cdot)$ is non-decreasing and $\int_0^t Y_i(s)\, dL_i(s) = 0$.
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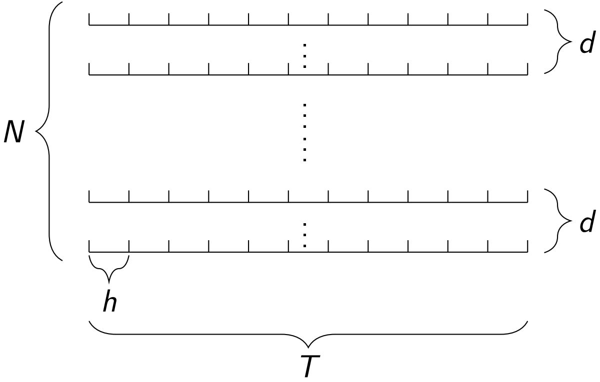
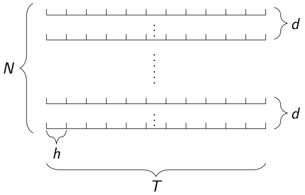
- Multi-dimensional Brownian motion $\mathbf{X} \to$ RBM $\mathbf{Y}$.
- Goal: Find an efficient simulation algorithm to estimate the steady-state expectation of certain functions $f(\cdot)$ of a general multi-dimension RBM for arbitrary dimension d.

A naive Euler scheme



- bias: $O(h)$;
- standard deviation: $O(N^{-1/2})$;
- Non-stationary error: $|f(\mathbf{Y}_T) - f(\mathbf{Y}_\infty)|$.

A naive Euler scheme: error analysis



- For a stepsize h, the number of independent sample path N, and horizon T:

$$\hat{\mathbf{Y}}(t+h) = \texttt{Skorokhod}(\hat{\mathbf{Y}}(t) + \Delta\mathbf{X}(t));$$

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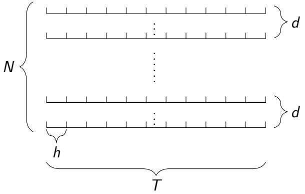

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- RMSE $\approx O(\sqrt{1/N + h^2})$.
- To make the RMSE $O(\epsilon)$, it requires $N = O(\epsilon^{-2})$ and $h = O(\epsilon)$. The computational complexity is $O(\epsilon^{-3}Td)$.

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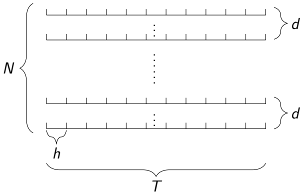

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- To make the RMSE $O(\epsilon)$, it requires $N = O(\epsilon^{-2})$ and $h = O(\epsilon)$. The computational complexity is $O(\epsilon^{-3}Td)$.
- What is the dependence of T on d and ϵ? And can we do better in terms of ϵ?

Multilevel Monte Carlo Algorithm

Assumptions

- **Uniform contraction:** let $R = I - Q^T$, where Q is substochastic and satisfies

$$\left\| \mathbf{1}^T Q^n \right\|_{\infty} \leq \kappa_0 (1 - \beta_0)^n, \;\; n \geq 1.$$

for $\beta_0 \in (0, 1)$ and $\kappa_0 \in (0, \infty)$ independent of d.

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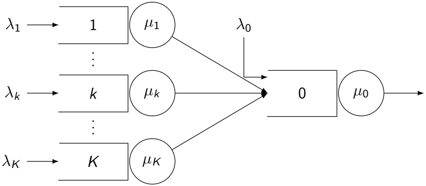
✗ A counter example:



$$Q = \begin{bmatrix} 0 & 0 & & 0 \\ 1 & \ddots & & \\ \vdots & & \ddots & \\ 1 & 0 & & 0 \end{bmatrix}$$

Assumptions: continue

- **Uniform stability:** let $\mathbf{X}(t) = \boldsymbol{\mu} t + C\mathbf{B}(t)$ and assume $R^{-1}\boldsymbol{\mu} < -\delta_0 \mathbf{1}$ for $\delta_0 > 0$ independent of d.

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- **Uniform marginal variability:** let $\Sigma = CC^T$ be the variance of the driven Brownian motion and assume $b_0^{-1} \leq \Sigma_{i,i} \leq b_0$ for $b_0 > 0$ independent of d.

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- **Lipschitz functions:** The function to be estimated $f(\cdot)$ is Lipschitz continuous in l_∞ norm, i.e. $|f(\mathbf{y}) - f(\mathbf{y}')| \leq \mathcal{L}\|\mathbf{y} - \mathbf{y}'\|_\infty$ for $\mathcal{L} > 0$ independent of d.
  - ✔ $f(\mathbf{y}) = \frac{1}{d}\sum_{i=1}^{d} y_i$;
  - ✔ $f(\mathbf{y}) = y_i$;
  - ✔ $f(\mathbf{y}) = \|\mathbf{y}\|_\infty$;
  - ✗ $f(\mathbf{y}) = \sum_{i=1}^{d} y_i$.

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- Parameters: discretization granularity $\gamma \in (0,1)$; simulation horizon $T > 0$; the total number of levels L.

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$$B_i^m(t) = B_i\left(t_m^-\right) + \left(t - t_m^-\right) \frac{B_i\left(t_m^+\right) - B_i\left(t_m^-\right)}{t_m^+ - t_m^-}, \text{ for } i = 1, 2, ..., d.$$

- $\mathbf{X}^m(t) = \boldsymbol{\mu} t + C\mathbf{B}^m(t)$.

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- $\mathbf{X}^m(t) = \boldsymbol{\mu} t + C\mathbf{B}^m(t)$.
- RBMs driven by $\mathbf{X}_{s:t}$ ($\mathbf{X}_{s:t}^m$) for $\mathbf{X}_{s:t}(u) = \mathbf{X}(s+u) - \mathbf{X}(s)$:

$$\begin{aligned} \mathbf{Y}\left(t+s; \mathbf{y}, \mathbf{X}_{0:s+t}\right) &= \mathbf{Y}\left(t; \mathbf{Y}\left(s; \mathbf{y}, \mathbf{X}_{0:s}\right), \mathbf{X}_{s:s+t}\right), \\ \mathbf{Y}^m\left(t+s; \mathbf{y}, \mathbf{X}_{0:s+t}^m\right) &= \mathbf{Y}^m\left(t; \mathbf{Y}^m\left(s; \mathbf{y}, \mathbf{X}_{0:s}^m\right), \mathbf{X}_{s:s+t}^m\right). \end{aligned} \quad (2)$$

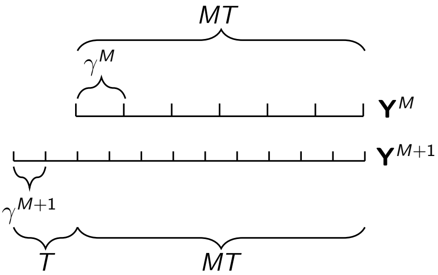
Multilevel Monte Carlo algorithm: estimator

Our estimator:

$$Z = \frac{1}{p(M)} \left(f\left(\mathbf{Y}^{M+1}\left(MT; \mathbf{Y}^{M+1}\left(T; \mathbf{y}_0, \mathbf{X}^{M+1}_{0:T} \right), \mathbf{X}^{M+1}_{T:(M+1)T} \right) \right) - f\left(\mathbf{Y}^{M}\left(MT; \mathbf{y}_0, \mathbf{X}^{M}_{T:(M+1)T} \right) \right) \right) + f\left(\mathbf{y}_0 \right).$$

for a random variable M following probability distribution

$$P(M=m) = p\left(m\right) = \gamma^m \left(1-\gamma\right)/(1-\gamma^L) \triangleq K(\gamma)\gamma^m, \text{ for } 0 \leq m < L.$$



Multilevel Monte Carlo algorithm: estimator

$$
\begin{aligned}
E[Z] &= E\left[E\left[Z|M\right]\right] \\
&= \sum_{m=0}^{L-1}\left(E\left[f\left(\mathbf{Y}^{m+1}\left(mT;\mathbf{Y}^{m+1}(T;\mathbf{y}_0,\mathbf{X}_{0:T}^{m+1}),\mathbf{X}_{T:(m+1)T}^{m+1}\right)\right)\right]\right. \\
&\quad \left.-E\left[f\left(\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}_{T:(m+1)T}^{m}\right)\right)\right]\right)+f\left(\mathbf{y}_0\right) \\
&= \sum_{m=0}^{L-1}\left(E\left[f\left(\mathbf{Y}^{m+1}\left((m+1)T;\mathbf{y}_0,\mathbf{X}_{0:(m+1)T}^{m+1}\right)\right)\right]\right. \\
&\quad \left.-E[f\left(\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}_{0:mT}^{m}\right)\right)]\right)+f\left(\mathbf{y}_0\right) \\
&= E\left[f\left(\mathbf{Y}^{L}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}^{L}\right)\right)\right].
\end{aligned}
$$

As $L\to\infty$,

$$E\left[f\left(\mathbf{Y}^{L}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}^{L}\right)\right)\right]\to E[f(\mathbf{Y}(\infty))].$$

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& -E\left[f\left(\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}_{T:(m+1)T}^{m}\right)\right)\right]\Bigg) + f\left(\mathbf{y}_0\right) \\
= & \sum_{m=0}^{L-1}\Bigg(E\left[f\left(\mathbf{Y}^{m+1}\left((m+1)T;\mathbf{y}_0,\mathbf{X}_{0:(m+1)T}^{m+1}\right)\right)\right] \\
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= & E\left[f\left(\mathbf{Y}^{L}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}^{L}\right)\right)\right].
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As $L \to \infty$,

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E\left[f\left(\mathbf{Y}^{L}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}^{L}\right)\right)\right] \to E[f(\mathbf{Y}(\infty))].
$$

Multilevel Monte Carlo Algorithm: Error Analysis

Key gradients to error analysis: discretization error

Lemma (Discretization Error)

For $0 < \gamma < 1$ and $m \geq 1$, let $\mathbf{X}^m(\cdot)$ be a discretized d-dimension Brownian path with step size γ^m. Then, we have for any $d \geq 2, m \geq 1$, $t > \gamma$,

$$E\left[\max_{1\leq i\leq d}\max_{0\leq s\leq t}(X_i^m(s) - X_i(s))^2\right] \leq C_0\gamma^m(\log(t)+\log(d)+m\log(1/\gamma)).$$

Lemma (Lipschitzness)

Suppose $\mathbf{Y}(t)$ and $\mathbf{Y}'(t) \in \mathbb{R}_+^d$ are the solutions to two Skorokhod problems with the same reflection matrix R, and input processes $\mathbf{X}(t)$ and $\mathbf{X}'(t)$ respectively for $t \in [0, T]$. Then,

$$\left|\mathbf{Y}(T) - \mathbf{Y}'(T)\right| \leq 2R \sup_{0\leq s\leq T}|\mathbf{X}(s) - \mathbf{X}'(s)|.$$

Key gradients to error analysis: non-stationary error

Lemma (Non-stationary Error)

There exist constants C_2 and $\xi_1 > 0$ such that

$$E[\|\mathbf{Y}(t;\mathbf{Y}(\infty),\mathbf{X}_{0:t}) - \mathbf{Y}(t;0,\mathbf{X}_{0:t})\|_\infty^2] \le C_2 d^3 \exp\left(-\xi_1 \frac{t}{\log(d)}\right).$$

Proof ideas: Directional derivative [Mandelbaum and Ramanan, 2010]:

$$\mathfrak{D}_{\mathbf{h}}\left(t;\mathbf{y},\mathbf{X}_{0:t}\right) = \lim_{\varepsilon\to 0} \frac{\mathbf{Y}\left(t;\mathbf{y}+\varepsilon\mathbf{h},\mathbf{X}_{0:t}\right) - \mathbf{Y}\left(t;\mathbf{y},\mathbf{X}_{0:t}\right)}{\varepsilon}, \forall\ \mathbf{h} \in \mathbb{R}^d.$$

After bounding $\mathfrak{D}_{\mathbf{h}}\left(t;\mathbf{y},\mathbf{X}_{0:t}\right)$ elementwise, we observe

$$\mathbf{Y}\left(t;\mathbf{y},\mathbf{X}_{0:t}\right) - \mathbf{Y}\left(t;0,\mathbf{X}_{0:t}\right) = \left(\int_0^1 \mathfrak{D}_{\mathbf{y}}\left(t;u\cdot\mathbf{y},\mathbf{X}_{0:t}\right)du\right).$$

The rest of proof are similar to Banerjee and Budhiraja [2019] and Blanchet and Chen [2020].

Error analysis: decomposition

The MSE of the estimator $\bar{Z} = \frac{1}{N}\sum_{i=1}^{N} Z_i$ is

$$
\begin{aligned}
& E[(\bar{Z} - E[f(\mathbf{Y}(\infty))])^2] \\
\leq \; & \frac{1}{N}\sum_{m=0}^{L-1} K(\gamma)^{-1}\gamma^{-m} V_m \\
+ \; & \left(E\left[f\left(\mathbf{Y}^L\left(TL; \mathbf{y}_0, \mathbf{X}^L_{0:LT}\right)\right)\right] - E[f(\mathbf{Y}(\infty))]\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
V_m = E\Bigg[\Bigg(& f\left(\mathbf{Y}^{m+1}\left((m+1)T; \mathbf{y}_0, \mathbf{X}^{m+1}_{0:(m+1)T}\right)\right) \\
& - f\left(\mathbf{Y}^{m}\left(mT; \mathbf{y}_0, \mathbf{X}^{m}_{T:(m+1)T}\right)\right)\Bigg)^2\Bigg].
\end{aligned}
$$

Error analysis: the variance term V_m

$$
\begin{aligned}
&\|\mathbf{Y}^{m+1}\left((m+1)T;\mathbf{y}_0,\mathbf{X}^{m+1}_{0:(m+1)T}\right)-\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}^{m}_{T:(m+1)T}\right)\|_\infty\\
&\le \|\mathbf{Y}^{m+1}\left((m+1)T;\mathbf{y}_0,\mathbf{X}^{m+1}_{0:(m+1)T}\right)-\mathbf{Y}\left((m+1)T;\mathbf{y}_0,\mathbf{X}_{0:(m+1)T}\right)\|_\infty\\
&+\|\mathbf{Y}^{m}\left(mT;\mathbf{y}_0,\mathbf{X}^{m}_{T:(m+1)T}\right)-\mathbf{Y}\left(mT;\mathbf{y}_0,\mathbf{X}_{T:(m+1)T}\right)\|_\infty\\
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&=\text{Discretization Errors}+\text{Non-stationarity Error.}
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&\le \left\|\mathbf{Y}^{m+1}\left((m+1)T;\mathbf{y}_0,\mathbf{X}^{m+1}_{0:(m+1)T}\right)-\mathbf{Y}\left((m+1)T;\mathbf{y}_0,\mathbf{X}_{0:(m+1)T}\right)\right\|_\infty \\
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Error analysis: the bias term

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& E\left[f\left(\mathbf{Y}^L\left(TL;\mathbf{y}_0,\mathbf{X}^L_{0:LT}\right)\right)\right]-E\left[f(\mathbf{Y}(\infty))\right] \\
= & \left(E\left[f\left(\mathbf{Y}^L\left(TL;\mathbf{y}_0,\mathbf{X}^L_{0:LT}\right)\right)\right]-E\left[f\left(\mathbf{Y}\left(TL;\mathbf{y}_0,\mathbf{X}_{0:LT}\right)\right)\right]\right) \\
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\end{aligned}
$$

Error bound

Parameter specification:

- Step size: we recommend γ around 0.05;
- Path length: $T = O(\log(d)^2)$;
- Number of levels: $L = \lceil (\log(\log(d)) + 2\log(1/\varepsilon) + k_1) / \log(1/\gamma) \rceil$;
- Number of sample paths:
  $N = \lceil (1-\gamma^L)(1-\gamma)^{-1}\gamma^{-L}L \rceil = O(\varepsilon^{-2}\log(d)\log(\log(d)))$.

Theorem

Suppose $\mathbf{Y}$ *(indexed by the number of dimensions d) is a sequence of RBMs satisfying Assumptions 1-4. Then, the total expected cost, in terms of* **the number of scalar Gaussian random variables**, *for the Multilevel Monte Carlo Algorithm to produce an estimator of* $E[f(\mathbf{Y}(\infty))]$ *with mean square error (MSE)* ε^2 *is*

$$O\left(\varepsilon^{-2} d \log(d)^3 (\log(\log(d)) + \log(1/\varepsilon))^3\right).$$

Numerical Results

Numerical experiments: setup

- Symmetric RBMs: $\boldsymbol{\mu} = -[1, 1, \ldots, 1]^T$

$$\Sigma = \begin{bmatrix} 1 & \rho_\sigma & \ldots & \rho_\sigma \\ \rho_\sigma & 1 & \ldots & \rho_\sigma \\ \vdots & & 1 & \vdots \\ \rho_\sigma & \ldots & \rho_\sigma & 1 \end{bmatrix}, \ R = \begin{bmatrix} 1 & -r & \ldots & -r \\ -r & 1 & \ldots & -r \\ \vdots & & 1 & \vdots \\ -r & \ldots & -r & 1 \end{bmatrix}.$$

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- Pick $\rho_\sigma = -\frac{1-\beta}{d-1}$ and $r = \frac{1-\beta}{d-1}$, and $f(Y(\infty)) = Y_1(\infty)$.

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- Pick $\rho_\sigma = -\frac{1-\beta}{d-1}$ and $r = \frac{1-\beta}{d-1}$, and $f(Y(\infty)) = Y_1(\infty)$.
- Closed form solution:

$$E[Y_1(\infty)] = \frac{1-(d-2)r+(d-1)r\rho_\sigma}{2(1+r)} = \frac{\beta}{2}.$$

Numerical experiments: setup

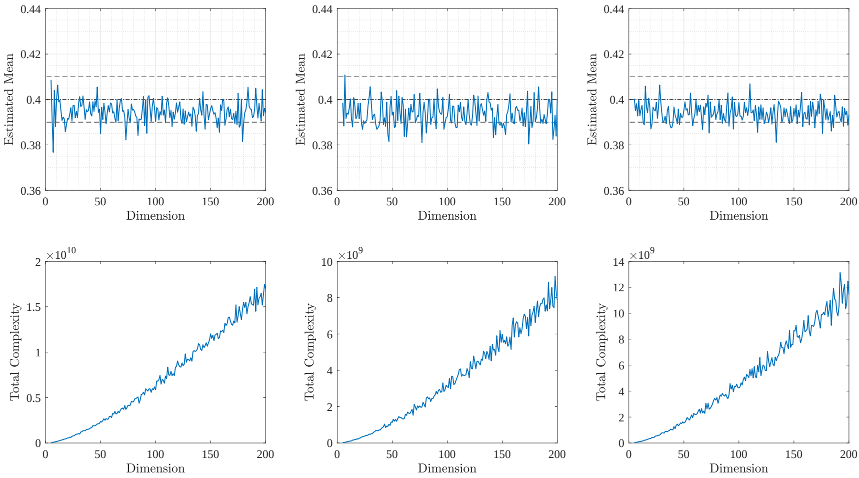
- Symmetric RBMs: $\boldsymbol{\mu} = -[1, 1, \ldots, 1]^T$

$$\Sigma = \begin{bmatrix} 1 & \rho_\sigma & \ldots & \rho_\sigma \\ \rho_\sigma & 1 & \ldots & \rho_\sigma \\ \vdots & & 1 & \vdots \\ \rho_\sigma & \ldots & \rho_\sigma & 1 \end{bmatrix}, \; R = \begin{bmatrix} 1 & -r & \ldots & -r \\ -r & 1 & \ldots & -r \\ \vdots & & 1 & \vdots \\ -r & \ldots & -r & 1 \end{bmatrix}.$$

- Pick $\rho_\sigma = -\frac{1-\beta}{d-1}$ and $r = \frac{1-\beta}{d-1}$, and $f(Y(\infty)) = Y_1(\infty)$.
- Closed form solution:

$$E[Y_1(\infty)] = \frac{1-(d-2)r + (d-1)r\rho_\sigma}{2(1+r)} = \frac{\beta}{2}.$$

- Pick $\beta = 0.8$ and $E[Y_1(\infty)] = 0.4$.



(a) $\gamma = 0.01$ (b) $\gamma = 0.05$ (c) $\gamma = 0.1$

Figure 1: Simulation results for symmetric RBMs at target error level $\epsilon = 0.01$.

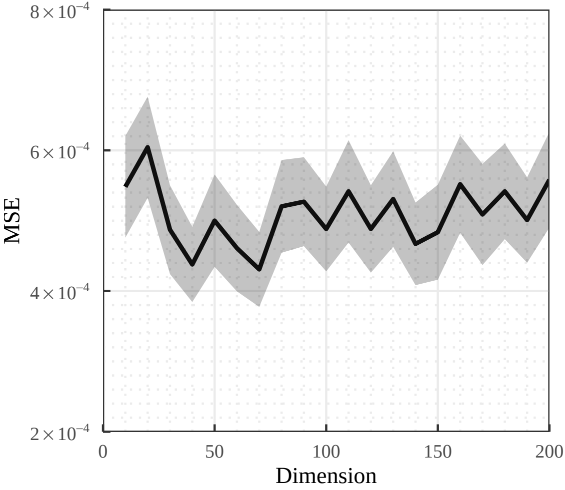


Figure 2: Mean square error of the estimators at target error level $\epsilon = 0.05$ for $\gamma = 0.05$. The shaded area represents 95% confidence band for the MSE.

Reference

Blanchent, Jose, Xinyun Chen, Peter Glynn, and **Nian Si**. "Efficient Steady-state Simulation of High-dimensional Stochastic Networks." *Stochastic Systems*, 11.2 (2021): 174-192.

Thanks!

Appendix: solve the Skorokhod problem

Algorithm 1 Algorithm for the Linear Complementarity Problem

Input:
The reflection matrix: R, and the initial vector: $\mathbf{x}$;
Output:
The solution of the linear complementarity problem: $\mathbf{y} \geq \mathbf{0}$, where $\mathbf{y} = \mathbf{x} + R\mathbf{L}$ for $\mathbf{L} \geq \mathbf{0}$.

Set $\epsilon = 10^{-8}$;
$\mathbf{y} = \mathbf{x}$;
while Exists $\mathbf{y}_i < -\epsilon$ **do**
    Compute the set $B = \{i : \mathbf{y}_i < \epsilon\}$;
    Compute $\mathbf{L}_B = -R_{B,B}^{-1}\mathbf{x}_B$;
    Compute $\mathbf{y} = \mathbf{x} + R_{:,B} \times \mathbf{L}_B$;
end while
return $\mathbf{y}$.

References

S. Banerjee and A. Budhiraja. Parameter and dimension dependence of convergence rates to stationarity for reflecting Brownian motions. *Working paper*, 2019.

J. Blanchet and X. Chen. Rates of convergence to stationarity for multidimensional RBM. *Mathematics of Operations Research, preprint*, 2020.

A. Mandelbaum and K. Ramanan. Directional derivatives of oblique reflection maps. *Mathematics of Operations Research*, 35:527–558, 2010.